Lecture 19

Modes of Convergence.

We have encountered different modes of conv. for seq. of fcns $f_n, f : X \to \mathbb{C}$, given measure sp. $(X, \mathcal{M}, \mu)$

(1) $f_n \to f$ uniformly

(2) $f_n \to f$ pointwise

(3) $f_n \to f$ a.e.

where (1) $\Rightarrow$ (2) $\Rightarrow$ (3) of course. We also have

(4) $f_n \to f$ in L^1 (i.e. $\|f - f_n\|_{L^1} \to 0$).

By themselves, neither of (1)–(3) $\Rightarrow$ (4), even if the $f_n, f \in L^1$, and (4) need not even imply (3), the weakest of (1)–(3). The purpose of this section in Folland is to sort out relations between these and add an additional mode.

Def. 1 $f_n \to f$ <u>in measure</u> if $\forall \varepsilon > 0$,

$$\mu(\{x : |f(x) - f_n(x)| > \varepsilon\}) \to 0 \text{ as}$$

$n \to \infty$.

Note. Easy to see that (1) can be formulated as $\forall \varepsilon > 0 \ \exists N: \{x : |f(x) - f_n(x)| > \varepsilon\} = \emptyset$ for $n \geq N$. Thus, (1) $\Rightarrow f_n \to f$ in measure. In general, (2) $\not\Rightarrow f_n \to f$ in measure.

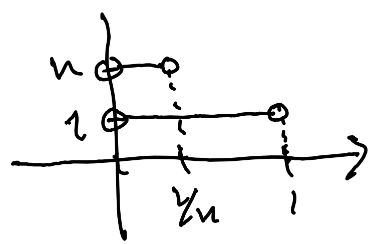
Prop1. $f_n \to f$ in $L^1 \Rightarrow f_n \to f$ in measure.

Pf. Suppose not $\Rightarrow \exists \varepsilon > 0, \delta > 0$, and subseq. $\{f_{n_k}\}$ s.t. $\mu(\{x : |f(x) - f_{n_k}(x)| > \varepsilon\}) > \delta$ $\forall k$. But then $\|f - f_{n_k}\| \geq \int_{\{|f - f_{n_k}| > \varepsilon\}} \varepsilon \, d\mu \geq \varepsilon\delta$

$\Rightarrow f_{n_k} \not\to f$ in L^1. $\Join$ $\blacksquare$

Ex1. Let $X = \mathbb{R}$, $\mu = m_1$ and consider

$$f_n(x) = n \, \chi_{(0, 1/n)}$$



Then, $f_n \to 0$ pointwise and $\{x : |f_n(x)| > \varepsilon\} = (0, 1/n) \Rightarrow f_n \to 0$ in measure, but as we have seen $f_n \not\to f$ in L^1.

However,

<u>Thm 1.</u> Let $\{f_n\}_{n=1}^{\infty}$ be Cauchy in measure.

(1) $\exists f$ measurable s.t. $f_n \to f$ in measure, and if $f_n \to g$ in measure, then $f = g$ a.e.

(2) $\exists$ subseq. $\{f_{n_k}\}$ s.t. $f_{n_k} \to f$ a.e.

<u>Pf:</u> Let $\varepsilon_k \downarrow 0$ (to be chosen). Cauchy in measure $\Rightarrow$ $\exists$ subseq. $\{n_k\}$ s.t. for $n, m \geq n_k$ $\mu(\{x : |f_n(x) - f_m(x)| > \varepsilon_k\}) < \varepsilon_k$.
Set $E_k = \{x : |f_{n_{k+1}}(x) - f_{n_k}(x)| > \varepsilon_k\} \Rightarrow$ $\mu(E_k) < \varepsilon_k$. Consider $F_\ell = \bigcup_{k=\ell}^{\infty} E_k$. If $x \notin F_\ell$, then $|f_{n_{k+1}}(x) - f_{n_k}(x)| \leq \varepsilon_k$ for $k \geq \ell$ $\Rightarrow$ $\forall\, i > j \geq \ell$,

$$|f_{n_i}(x) - f_{n_j}(x)| \leq \sum_{m=j}^{i-1} |f_{n_{m+1}}(x) - f_{n_m}(x)| \leq$$

$$\sum_{m=j}^{\infty} \varepsilon_m .$$

and also

$$\mu(F_\ell) \le \sum_{m=\ell}^{\infty} \varepsilon_m .$$

Now, choose $\varepsilon_k = 2^{-k} \Rightarrow \sum_{m=\ell}^{\infty} \varepsilon_m = 2^{1-\ell}$

$\Rightarrow \mu(F_\ell) \to 0$. Moreover, on F_ℓ^c the seq. $\{f_{n_k}\}$ is uniformly Cauchy and, in particular, pointwise Cauchy. Moreover $F_\ell \downarrow$. Setting $F = \bigcap_{\ell=1}^{\infty} F_\ell$, we find that on $F^c = \bigcup_{\ell=1}^{\infty} F_\ell^c$ $\{f_{n_k}\}$ converges pointwise. Since $\mathbb{C}$ is complete, we obtain a function $f : F^c \to \mathbb{C}$ s.t. $f_{n_k} \to f$ pointwise on F^c. We also note that $\mu(F) = \lim_{\ell \to \infty} \mu(F_\ell)$ $= 0$ by cont. from above.

If we extend f to X by setting $f=0$ on the nullset F, then $f_{n_k} \to f$ a.e. and f is <u>meas.</u> since $\chi_{F^c} f_{n_k} \to f$ everywhere.

<u>Claim 1</u> $f_{n_k} \to f$ in measure.

Pick $\varepsilon > 0$ and k s.t. $2^{-k+1} < \varepsilon$. Then,

$$x \in F_k^c \Rightarrow |f(x) - f_{n_k}(x)| \le \underbrace{|f(x) - f_{n_m}(x)|}_{\to 0 \text{ as } m \to \infty} +$$

$$+ |f_{n_m}(x) - f_{n_k}(x)| \le 2^{-k+1}$$

$$\Rightarrow \{x : |f(x) - f_{n_k}(x)| > \varepsilon\} \subseteq F_k \Rightarrow$$

$$\mu(\{ \quad \}) \le \mu(F_k) \le 2^{-k+1} \to 0,$$

which proves the claim.

<u>Claim 2.</u> $f_n \to f$ in measure.

Pick $\varepsilon > 0$ and k s.t. $2^{-k+1} < \varepsilon/2$

If now $\quad |f(x) - f_n(x)| > \varepsilon,$

then $\varepsilon < |f(x)-f_n(x)| \le |f(x)-f_{n_k}(x)| + |f_{n_k}(x)-f_n(x)|$ $\Rightarrow$ at least one of $|f(x)-f_{n_k}(x)| > \varepsilon/2$ or $|f_{n_k}(x)-f_n(x)| > \varepsilon/2$

Thus, $\{x : |f(x)-f_n(x)| > \varepsilon\} \subseteq F_k \cup \{x : |f_{n_k}(x) - f_n(x)| > \varepsilon/2\}$. Choose $\delta > 0$ and N_δ s.t. $\mu(\{x : |f_n(x)-f_m(x)| > \varepsilon/2) < \delta/2$.

If we increase k s.t. $n_k \geq N_\delta$ and $2^{-k+1} < \min(\frac{\delta}{2}, \varepsilon/2)$, then we conclude that for $n \geq N_\delta$

$$\mu(\{x : |f(x)-f_n(x)| > \varepsilon\} \le \mu(F_k) + \delta/2$$
$$\le 2^{-k+1} + \frac{\delta}{2} < \delta,$$

Thus, $\mu(\{|f-f_n| > \varepsilon\}) \to 0$ as $n \to \infty$.

Finally, we note that if $f_n \to g$ also in measure, then the

the subseq. $\{f_{n_k}\}$ converges in measure to g as well. But then there is a subseq. $\{f_{n_{k_\ell}}\}$ of the subseq. s.t. $f_{n_{k_\ell}} \to g$ a.e. Since $f_{n_{k_\ell}} \to f$ a.e. as well, we conclude that $f = g$ outside the union of the two nullsets N_g, N_f outside of which $f_{n_{k_\ell}}$ converges to g and f respectively. Thus, $f = g$ a.e. as claimed. $\boxtimes$

A direct corollary of Prop 2 + Thm 1:

Cor 1. If $f_n \to f$ in L^1, then $\exists$ subseq. $\{f_{n_k}\}$ s.t. $f_{n_k} \to f$ a.e.